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The Six and Eight-Vertex Models Revisited

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Elliott Lieb's ice-type models opened up the whole field of solvable models in statistical mechanics. Here we discuss the "commuting transfer matrix" T, Q equations for these models, writing them in a more explicit and transparent notation that we believe offers new insights. The approach manifests the relationship between the six-vertex and chiral Potts models, and between the eight-vertex and Kashiwara–Miwa models.

KEY WORDS: Statistical mechanics; lattice models; ice-models; six-vertex model.

1. INTRODUCTION

Elliott Lieb used the Bethe ansatz to solve the two-dimensional ice, F and KDP models in 1967—all typical cases of the general "six-vertex" model whose solution was then given by Sutherland.[1–5] This work, together with Onsager's famous solution[6] of the square-lattice Ising model in 1944, formed the basis on which the field of two-dimensional models rapidly grew. It led the author to develop the "commuting transfer matrix" of tackling such problems, notably the eight-vertex and hard-hexagon models.

This method begins with two steps. The first is to treat the problem in sufficient generality that it contains an arbitrary parameter (a complex number) v such that the transfer matrix T is a non-trivial function $T(v)$ of v, and

$$T(u)\,T(v) = T(v)\,T(u) \tag{1}$$

for all values of u and v. Thus $T(u)$ and $T(v)$ commute. The variable v is known as the "spectral parameter:" for the six and eight-vertex models

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(and many other planar models) it is the difference of two "rapidity" variables associated with the horizontal and vertical directions of the lattice.

The second step is to construct (if only implicitly) another matrix function $Q(u)$ which also commutes with $T(v)$, i.e.,

$$Q(u)\,T(v) = T(v)\,Q(u), \tag{2}$$

and satisfies a matrix functional relation that is linear in T and Q, and homogeneous in Q. For the six-vertex model this has the structure

$$T(v)\,Q(v) = \phi(v-\lambda)\,Q(v+2\lambda') + \phi(v+\lambda)\,Q(v-2\lambda'), \tag{3}$$

$\phi(v)$ being a known scalar function and λ' a "crossing parameter."

A further step is to show that $Q(u)$ and $Q(v)$ commute:

$$Q(u)\,Q(v) = Q(v)\,Q(u), \tag{4}$$

for all u, v.

The final step is to go to a representation in which $T(v)$, $Q(v)$ are diagonal matrices, for all v. Then (3) becomes a scalar functional relation for each eigenvalue, which can be solved.

For the six and eight-vertex models, all these steps have been carried out by the author[7–9] and written up in chapters 9 and 10 of ref. 10. However, the construction of $Q(v)$ given therein is implicit rather than explicit.

Further, in the early papers[7–9] the author was concerned to focus on cases where there was no reason to believe the matrix $Q(v)$ was in general singular. Obviously if it is zero the equations (2)–(4) contain no information on $T(v)$. Only if it is non-singular (i.e., has non-zero determinant) for general values of v can one expect to obtain all the eigenvalues of $T(v)$ from (3). For this reason the author initially focussed on the "roots of unity" cases, where one can write $Q(v)$ more explicitly. Later he was able to remove this restriction, and the working in ref. 10 is a quite general solution of the zero-field six and eight-vertex models.

Here we re–present this working for the six-vertex model, giving a more explicit expression for the matrix $Q(v)$.[2] The relations (1)–(4) are derived as the set of equations (58), where the skew parameter r is exhibited and (for L even) the other skew or field parameter s can be given the value 0.

[2] Basically our results are more explicit versions of equations (9.8.15)–(9.8.37) of ref. 10.

The formulae (27), (29) for the transfer matrix $Q_R(v)$ are interesting in that it is almost that of a trivial one-dimensional model: the only thing that stops this happening is a factor $x^{d(a-b)}$ in the Boltzmann weight function (27). A very similar property occurs in the three-dimensional Zamolodchikov model and its extension.[11, 12]

In 1990 Bazhanov and Stroganov[13] showed how the recently-discovered solvable chiral Potts model could be obtained from the six-vertex model via the matrix $Q(v)$. They used the general Q matrix discussed in Section 6 (also including the fields we mention at the end of our conclusion).[3] In this approach the Boltzmann weight of the chiral Potts model first appears as an auxiliary function that enters the derivation of (4). Here we shall observe such a property.

We emphasize that our working is for *all* values of the crossing parameter λ or λ'. This means that the spins in our spin formulations of the six-vertex and Q models are free to take all integer values from $-\infty$ to $+\infty$. We use this convenient "spin language" to derive the relations, but then in Section 5 we indicate how one can transform back to the arrow language of Lieb so as to ensure that the row-to-row transfer matrices are finite-dimensional.

Only in the conclusion do we address the "roots of unity" cases, when $\lambda' = \mathrm{i}m\pi/N$, $x^{4N} = 1$, (m, N integers) and the spin on a given site takes just N states. These cases are of course of interest. Fabricius and McCoy[15–18] have studied these cases and have emphasized that the eigenvalues of $T(v)$ are then degenerate.[4] This provides a motivation for establishing the Q, Q commutation relations directly: if the eigenvalues of $T^r(v)$ are non-degenerate, the Q, Q relations are implied by those for T, T and T, Q. Further, it is this step that first introduces the weight function of the chiral Potts model. Only for the N-state root-of-unity case does this model seem to be properly defined.

Finally, we briefly indicate how the same approach applied to the eight-vertex model leads quite directly to the Kashiwara–Miwa model.[22] As Hasegawa and Yamada have shown,[23] the Kashiwara–Miwa model is a "descendant" of the zero-field eight-vertex model in the same way that the chiral Potts model is a "descendant" of the six-vertex model in external fields.

[3] It is the $\tau_2(t_q)$ matrix of ref. 14, except that it is rotated through 90°. Bazhanov and Stroganov rotated the lattice and started with the column-to-column transfer matrix of the six-vertex model. This leads to the usual row-to-row transfer matrix of the chiral Potts model. Here we shall not be extending the calculation that far, so revert to using the row-to-row transfer for the six-vertex model.

[4] Fabricius and McCoy use the word "incomplete" in a way quite different from its usual sense in mathematical physics.[19–21]

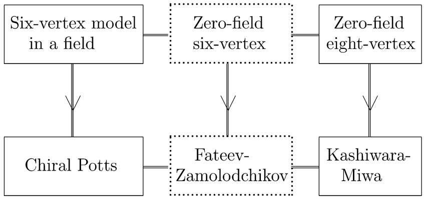

Fig. 1. The three vertex models and their respective "descendants." The central models are special cases of the ones to their left and to their right.

This answers a question that has long puzzled the author: can one obtain a generalization of chiral Potts by using the Bazhanov–Stroganov method, but starting with the eight-vertex model instead of the six-vertex? The answer appears to be no. The model from which Bazhanov and Stroganov start is the six-vertex model in a field. Hasegawa and Yamada start from the zero-field eight-vertex model. There is no known solvable model that continuously generalizes and includes both these vertex models, so there is no reason to suppose that there exists a model which similarly generalizes and includes the chiral Potts and Kashiwara–Miwa models.

It is true that the two vertex models intersect in the zero-field six-vertex model. The "descendant" of this is the Fateev–Zamodochikov model,[24] which is indeed the intersection of the chiral Potts and Kashiwara–Miwa models.

The relationships between these six models are shown in Fig. 1.

2. TRANSFER MATRICES AND THE STAR-TRIANGLE RELATION

Here we make some general observations to which we shall refer in subsequent sections.

We shall consider various "interactions-round-a-face" (IRF) models on the square lattice $\mathscr{L}$ of $\mathscr{N}$ sites. In each model, each site of the lattice carries a "spin." Let a, b, c, d be the four spins round a face, arranged as in Fig. 2, and let $W(a, b, c, d)$ be the Boltzmann weight of this spin configuration. Then the partition function is

$$Z = \sum \prod W(a, b, c, d), \qquad (5)$$

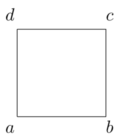

Fig. 2. Four spins round a face of the square lattice.

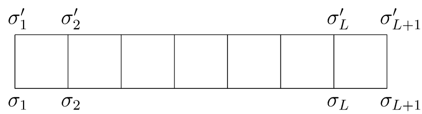

Fig. 3. A typical row of L faces of the square lattice, with spins $\sigma_1,\ldots,\sigma_{L+1}$ on the lower row of sites, $\sigma'_1,\ldots,\sigma'_{L+1}$ on the upper.

the product being over all faces and the sum over all allowed values of all the $\mathcal{N}$ spins.[5] There are usually restrictions on the spin values: here we shall require that on *some* (not necessarily all) edges (i, j) the two adjacent spins σ_i, σ_j differ by unity:

$$\sigma_j = \sigma_i \pm 1. \tag{6}$$

In particular, we require that (6) be satisfied on all horizontal edges. Let the lattice have L columns, i.e., L faces per row, and let the spins in any particular row be $\sigma_1, \sigma_2,\ldots, \sigma_{L+1}$. Then we impose skewed cyclic boundary conditions:

$$\sigma_{L+1} = \sigma_1 + r, \tag{7}$$

where r is the "skew parameter"—an integer with value between $-L$ and L such that $L-r$ is even. It may vary from row to row.

We shall also use the spin differences:

$$\alpha_i = \sigma_{i+1} - \sigma_i. \tag{8}$$

We see that each α_i takes the values $+1$ or -1, and

$$\alpha_1 + \cdots + \alpha_L = r. \tag{9}$$

Let $\sigma = \{\sigma_1,\ldots, \sigma_{L+1}\}$ be the set of spins in one row, and let $\sigma' = \{\sigma'_1,\ldots, \sigma'_{L+1}\}$ be the spins in the row above, as in Fig. 3. Then the row-to-row *transfer matrix* T is the matrix with elements

$$T_{\sigma,\sigma'} = W(\sigma_1, \sigma_2, \sigma'_2, \sigma'_1)\, W(\sigma_2, \sigma_3, \sigma'_3, \sigma'_2) \cdots W(\sigma_L, \sigma_{L+1}, \sigma'_{L+1}, \sigma'_L). \tag{10}$$

[5] Some of the spins may need to be fixed to avoid this sum being infinite. For the six-vertex model just one spin on the lattice needs to be fixed—say the one at the bottom-right corner. For the "Q" model we shall discuss we need to fix one spin in every row - say the left-most spin.

If the lattice has M rows, with cyclic top-to-bottom boundary conditions, it follows that

$$Z = \text{Trace}\, T^M. \tag{11}$$

Obviously T depends on the skew parameter r. If r is the same for both rows, we may write T as T^r. If it is different, so that $\sigma_{L+1} = \sigma_1 + r$ and $\sigma'_{L+1} = \sigma'_1 + s$, we may write T as T^{rs}.

Commutation

Now consider two models, with different weight functions W_1 and W_2. Let their transfer matrices be T_1, T_2 (for the moment we suppress the superfixes r, s). The product $T_1 T_2$ has entries

$$[T_1 T_2]_{a,b} = \text{Trace}\, M_1 M_2 \cdots M_L. \tag{12}$$

Here M_j is a a matrix which depends on four outer spins $a_j, a_{j+1}, b_j, b_{j+1}$ and has entries

$$[M_j]_{c,d} = W_1(a_j, a_{j+1}, d, c)\, W_2(c, d, b_{j+1}, b_j). \tag{13}$$

The entries of $T_2 T_1$ are similar, but M_j is replaced by M'_j, in which W_1 and W_2 are interchanged. For T_1 to commute with T_2, i.e., for $T_1 T_2 = T_2 T_1$, we need there to exist invertible matrices $P_1, \ldots, P_{L+1}$ such that P_j depends on the outer spins a_j, b_j (and no others), and

$$M'_j = P_j^{-1} M_j P_{j+1}, \qquad j = 1, \ldots, L. \tag{14}$$

Remembering that the matrix P_j depends on a_j, b_j, write its entry (c, d) as $W_3(c, a_j, d, b_j)$. Then, rewriting (14) as $M_j P_{j+1} = P_j M'_j$, we obtain this condition explicitly as

$$\begin{aligned}
&\sum_g W_1(b, c, g, a)\, W_2(a, g, e, f)\, W_3(g, c, d, e) \\
&\qquad = \sum_g W_3(a, b, g, f)\, W_2(b, c, d, g)\, W_1(g, d, e, f)
\end{aligned} \tag{15}$$

for all allowed values of the external spins a, b, c, d, e, f.

For interaction-round-a-face models, (15) is the "star-triangle" relation. It is depicted graphically in Fig. 4.

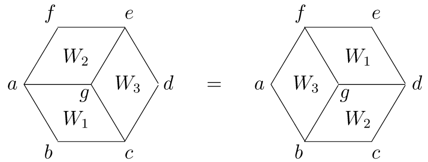

Fig. 4. The generalized star-triangle relation.

For cyclic boundary conditions, $P_{L+1} = P_1$ and (15), together with the invertibility of P_j, is sufficient to ensure that $T_1 T_2 = T_2 T_1$, i.e., T_1, T_2 commute.

However, for our more general skewed boundary conditions we do have a problem. Including the skew parameters, the commutation relation in general becomes

$$T_1^{rt} T_2^{ts} = T_2^{rt'} T_1^{t's}. \tag{16}$$

Here r is the skew parameter of the lowest row, s of the uppermost, and t, t' of the intervening rows (the ones whose spins are summed over in the matrix multiplication). Note that t is not necessarily the same as t'. If P_1 has entries $W_3(c, a_1, d, b_1)$, then P_{L+1} has entries $W_3(c+t, a_1+r, d+t', b_1+s)$. If these entries are equal then (16) holds true.

In fact they are not always equal: in Section 4 we encounter a case where

$$W_3(c+t, a+r, d+t', b+s) = x^{s(c-a)} W_3(c, a, d, b), \tag{17}$$

x being the fixed parameter defined in (31). The effect of this is to insert an extra factor $x^{s(c_1-a_1)}$ into the sum over $c_1, \ldots, c_L$ that is the matrix product $T_1 T_2$. This is equivalent to replacing T_1 therein by $D^{-s} T_1 D^s$, where D is a a diagonal matrix with entries

$$D_{\sigma,\sigma'} = x^{\sigma_1} \delta(\sigma_1, \sigma_1') \cdots \delta(\sigma_{L+1}, \sigma'_{L+1}). \tag{18}$$

(Note that here σ and σ' necessarily have the same value of the skew parameter $\sigma_{L+1} - \sigma_1$.)

Thus instead of (16) we obtain

$$D^{-s} T_1^{rt} D^s T_2^{ts} = T_2^{rt'} T_1^{t's}. \tag{19}$$

3. THE SIX-VERTEX MODEL

As considered by Lieb and Sutherland, the six-vertex model is a model on the square lattice where one puts an arrow on every edge. At each site

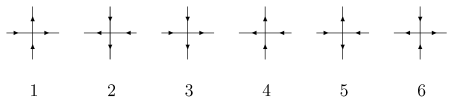

Fig. 5. The six arrow configurations of the six-vertex model.

one is required to satisfy the "ice rule" that there be two arrows pointing in and two pointing out of the site. There are then six possible configurations of arrows at a site, as indicated in Fig. 5.

For our present purposes, we wish to express this in interaction-round-a-face language. We do this by putting spins on the faces of the lattice so that as one goes from one face to a neighbouring face, if the intervening arrow points to one's left (right), then the spin value increases (decreases) by unity.

As one walks around a site, the ice rule ensures that there are two arrows pointing to the left, and two to the right, so one does indeed return to the original spin value.

It follows that a spin configuration is allowed iff every pair of adjacent spins (horizontal *and* vertical) differs by unity, i.e., satisfies (6). If one of the spins is fixed (say the one at the lower-right-hand corner), then there is a one-to-one correspondence between allowed arrow and allowed spin configurations on the lattice.

Finally, we go to the dual of the arrow lattice. The spins now live on sites (rather than faces), and Fig. 5 becomes Fig. 6.

The spin differences α_i mentioned above now describe the vertical arrows in a row: $\alpha_i = +1$ if the arrow between spins σ_i and σ_{i+1} is pointing upwards, $\alpha_i = -1$ if it is pointing downwards. We identify the horizontal arrow between σ_1 and σ'_1 with the arrow between σ_{L+1} and σ'_{L+1} (i.e., in the arrow formulation we use the usual cyclic boundary conditions). Then the ice rule implies that if (9) is satisfied for one row, then it is satisfied for all rows, with the *same* value of r. This is the "conservation of up and down arrows" property of the six-vertex model.[1–5]

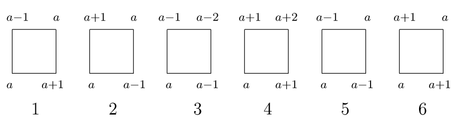

Fig. 6. The six configurations of the six-vertex model, expressed in terms of spins on the dual lattice.

Let $w_1, \ldots, w_6$ be the Boltzmann weights of the six configurations in Figs. 5 and 6. Here we consider only the "zero-field" six-vertex model, which is invariant under reversal of all arrows, so we take:

$$w_1 = w_2, \qquad w_3 = w_4, \qquad w_5 = w_6. \tag{20}$$

The overall normalization of w_1, w_2, w_3 plays a trivial role in the calculations, so without loss of generality we can define two parameters λ, v so that

$$w_2 = \sinh \tfrac{1}{2}(\lambda - v), \qquad w_4 = \sinh \tfrac{1}{2}(\lambda + v), \qquad w_6 = \sinh \lambda \tag{21}$$

as in (9.2.3) of ref. 10. It follows that the Boltzmann weight function of the zero-field six vertex model is

$$\begin{aligned} W(a, b, c, d) = W_{6v}(v \,|\, a, b, c, d) = \delta(a, c)\, \mathrm{e}^{(\lambda+v)(b+d-a-c)/4} \sinh \tfrac{1}{2}(\lambda - v) \\ + \delta(b, d)\, \mathrm{e}^{(\lambda-v)(a+c-b-d)/4} \sinh \tfrac{1}{2}(\lambda + v). \end{aligned} \tag{22}$$

We regard λ (the "crossing parameter") as a given constant and v as a variable (the "spectral parameter"). We therefore write the six-vertex model transfer matrix as $T(v) = T^r(v)$ and define it as in (10):

$$[T^r(v)]_{\sigma, \sigma'} = W_{6v}(v \,|\, \sigma_1, \sigma_2, \sigma'_2, \sigma'_1) \cdots W_{6v}(v \,|\, \sigma_L, \sigma_{L+1}, \sigma'_{L+1}, \sigma'_L). \tag{23}$$

The lower and upper spin sets σ, σ' have the same value of the skew parameter r.

The star-triangle relation (15) is satisfied if

$$W_1 = W_{6v}(v), \qquad W_2 = W_{6v}(v'), \qquad W_3 = W_{6v}(v' - v - \lambda)$$

for all values of v and v'. Also, taking $r, s, s', t = r$, we see that (17) is satisfied by (22), with $x = 1$.

The six-vertex model transfer matrices therefore commute:

$$T^r(v)\, T^r(v') = T^r(v')\, T^r(v). \tag{24}$$

Multiplying $W_{6v}(a, b, c, d)$ by a field factor μ^{c-b} is equivalent to post-multiplying $T^r(v)$ by a diagonal matrix S^z with entries

$$S^z_{\sigma, \sigma'} = \zeta(\sigma) \prod_{j=1}^{L+1} \delta(\sigma_j, \sigma'_j), \tag{25}$$

where

$$\zeta(\sigma) = \prod_{j=1}^{L} \mu^{\sigma_{j+1}-\sigma_j} = \mu^r.$$

This in turn corresponds to introducing an electric field acting on the upper vertical arrows, or equivalently on the upper pairs of horizontally adjacent spins. Clearly S^z commutes with $T^r(v)$:

$$S^z\, T^r(v) = T^r(v)\, S^z. \tag{26}$$

4. THE "*Q*" MODEL

The step taken by Bazhanov and Stroganov[13] was to look for another solution of (15), in which W_3 remains unchanged, but W_1, W_2 are altered so that they are no longer weight functions for the six-vertex model, but for a new "Q"-model. These functions can in fact be obtained (after changing the normalization) from (9.8.15)–(9.8.23) of ref. 10. They are

$$W_Q(v \mid a, b, c, d) = \exp[\lambda' d(a-b)/2 - (\lambda+v)(b-a)(c-d)/4], \tag{27}$$

where

$$\lambda' = \lambda - \mathrm{i}\pi, \tag{28}$$

as in Eq. (9.3.7) of ref. 10.

Horizontally adjacent spins must still satisfy (6), so $|b-a| = |c-d| = 1$, but now there are no restrictions on the differences $d-a$ or $c-b$ of vertically adjacent spins. Hence the skew parameters r, s of the lower and upper rows in Fig. 3 can be different.

We define the corresponding transfer matrix $Q_R^{rs}(v)$ as in (10):

$$[Q_R^{rs}(v)]_{\sigma,\sigma'} = W_Q(v \mid \sigma_1, \sigma_2, \sigma_2', \sigma_1') \cdots W_Q(v \mid \sigma_L, \sigma_{L+1}, \sigma_{L+1}', \sigma_L'). \tag{29}$$

T, *Q* Commutation.

In the star-triangle relation (15) we now substitute

$$W_1 = W_{6v}(v), \qquad W_2 = W_Q(v'), \qquad W_3 = W_Q(v'-v-\lambda). \tag{30}$$

To be consistent with the definitions of W_{6v} and W_Q, we require that spins linked directly by edges in Fig. 4 differ by one, *except* for the edges (a, f),

(g, e), (c, d) on the lhs, and edges (a, f), (b, g), (c, d) on the rhs. This means that the six external spins break up into two sets: (a, b, c) and (d, e, f). One is free to independently increment all the spins in either set by unity. We find that (15) is satisfied by (30).

For the T and Q_R matrices to commute, we also need to check the the auxiliary condition (17). Since T necessarily relates two rows of spins with the same skew parameter, we must have $t = r$, $t' = s$ therein. We find that (17) is satisfied, but x therein is *not* unity: instead

$$x = \mathrm{e}^{\lambda'/2} = -\mathrm{i}\mathrm{e}^{\lambda/2}. \tag{31}$$

We therefore obtain the modified commutation relation (19), with T_1 replaced by $T^r(v)$ and T_2 by $Q_R^{rs}(v')$, i.e.,

$$D^{-s}\, T^r(v)\, D^s\, Q_R^{rs}(v') = Q_R^{rs}(v')\, T^s(v). \tag{32}$$

Here v and v' are arbitrary, D is defined by (18) and (31). The notation is threatening to become confusing: note that r and s here are merely superfixes, except that D^s is actually the diagonal matrix D raised to the power s.

The *T*, *Q* Functional Relation.

Our working in Section 2 depended on the matrices P_j being invertible. This will cease to be so if $W_3(a, b, c, d)$ factors into a function independent of a times a function independent of c. For the derivation of (32) this happens when $v' = v$.

This does not mean that the commutation relation fails—it follows by taking the limit $v' \to v$. It does mean that there is then additional information in the relation (15). The lhs then involves d only via the simple factor $x^{e(g-c)}$, while the rhs involves a only via $x^{f(a-b)}$.

Going back to equations (12)–(14), define vectors $\xi_1,\ldots,\xi_L$ with elements

$$[\xi_j]_c = x^{b_j(c-a_j)}. \tag{33}$$

Then from the above observations about the star-triangle relation

$$M_j \xi_{j+1} = h_j \xi_j, \tag{34}$$

where

$$h_j = h(a_j, a_{j+1}, b_{j+1}, b_j)$$

is a scalar factor dependent (as are M_j and ξ_j) on the lower and upper spins $a_j, a_{j+1}, b_j, b_{j+1}$. In fact

$$h(a, b, c, d) = \sum_g W_{6v}(v \mid a, b, g, f)\, W_Q(v \mid f, g, c, d)\, x^{c(g-b)-d(f-a)}. \quad (35)$$

Here W_{6v}, W_Q are defined by (22), (27). The rhs is necessarily independent of the extra spin f.

A direct calculation of (35) reveals that

$$h(a, b, c, d) = \sinh \tfrac{1}{2}(\lambda - v)\, W_Q(v + 2\lambda' \mid a, b, c, d) \quad (36)$$

Since W_1 in (13) is the six-vertex model weight function, c is $a_j \pm 1$ and d is $a_{j+1} \pm 1$. Hence each M_j is a two-by-two matrix. Now take P_j to be a two-by-two matrix whose first column is ξ_j. The second column can be chosen arbitrarily, so long as the choice makes P_j invertible. It is convenient to choose $\det(P_j) = 1$.

Then (34) ensures that

$$P_j^{-1} M_j P_{j+1} = \tilde{M}_j, \quad (37)$$

where $\tilde{M}_j$ is an upper-triangular two-by-two matrix with upper-left entry h_j. Equating determinants, we find that the lower-right entry of $\tilde{M}_j$ is $h'(a_j, a_{j+1}, b_{j+1}, b_j)$, where

$$h'(a, b, c, d) = \sinh \tfrac{1}{2}(\lambda + v)\, W_Q(v - 2\lambda' \mid a, b, c, d). \quad (38)$$

Again, we have to worry about our skewed boundary conditions. In general we do not have $P_{L+1} = P_1$, but rather $P_{L+1} = \mathscr{D} P_1$, where $\mathscr{D}$ is a diagonal two-by-two matrix with entries $x^{t(c-a_1)} \delta(c, d)$. As in (19), we must therefore replace T_1 by $D^{-s} T_1 D^s$, which is equivalent to replacing the rhs of (12) by Trace $\mathscr{D} M_1 \cdots M_L$. Then the similarity transformation (37) reduces the matrix product to upper-triangular form, so the trace becomes the sum of two products, the first over all upper-left elements of the $\tilde{M}_j$, the second over all lower-right. We obtain

$$D^{-s}\, T^r(v)\, D^s\, Q_R^{rs}(v) = \phi(\lambda - v)\, Q_R^{rs}(v + 2\lambda') + \phi(\lambda + v)\, Q_R^{rs}(v - 2\lambda'), \quad (39)$$

where

$$\phi(v) = [\sin(v)]^L. \quad (40)$$

Q, Q Commutation

Now we consider the conditions for two transfer matrices $Q_R^{rs}(v)$, with different values of v and possibly different values of r, s, to commute. This is the step at which one first encounters the chiral Potts model.

As we have seen, the six-vertex model transfer matrices T satisfy the commutation relation (24), and the Q matrices satisfy the commutation relation (32) with T. At first sight this would appear to imply a corresponding commutation relation between the Q's themselves. If the T have unique eigenvalues, this is so.

However, Fabricius and McCoy have emphasized that for the special case when our parameter x is a complex root of unity, then the eigenvalues of the T matrices are degenerate, and interesting algebraic structures emerge.[6] It is therefore desirable to establish the Q, Q commutation relations directly.

We immediately strike a difficulty with our spin language. Up to now we have only considered matrix products that involve at least one six-vertex T matrix, e.g., TQ_R. Each element of this product corresponds to two rows of faces of the lattice, the lower with transfer matrix T, the upper with Q_R. There are three rows of spins on sites, those in the lowest and uppermost are given, the ones in the middle row are to be summed over. Since at least one of the matrices is T, the ice rule (all horizontally *and* vertically adjacent spins differ by one) ensures that that there are a finite number of terms in this summation.

However, if both matrices are Q_R matrices, one can choose one of the spins in the middle row arbitrarily. At this stage we are allowing spins to take all integer values, so there are an infinite number of choices and an infinite number of terms in the summation. The matrix product is not defined.

To overcome this, we anticipate the next section, where we go from spin language back to "horizontal spin difference" or "vertical arrow" language. If we can arrange a QQ product so that the terms in the sum it represents are unchanged by incrementing all the spins in the middle row by unity, then we can write this sum as a sum over only the differences of such spins. This is equivalent to arbitrarily fixing one of the spins in the middle row, say the left-hand spin.

From (27) and (29), incrementing by unity all the spins in the lower row of the matrix Q_R^{rs} leaves the elements of that matrix unchanged. It is

[6] The eigenvectors of T are then of course not unique, so it is not surprising that (39) then defines the eigenvalues of T uniquely, but not those of Q—a fact that Fabricius and McCoy seem to find remarkable.

not so the upper row: that introduces a factor x^{-r} coming from the $\exp[\lambda' d(a-b)/2]$ term in (27). We can overcome this by instead using the matrix

$$Q_L^{rs}(v) = x^{rs} D^s Q_R^{rs}(v)\, D^r \tag{41}$$

which has an extra factor $x^{r\sigma'_1 + s\sigma_1 + rs}$ in (29). This is in turn equivalent to multiplying (27) by x^{bc-ad}, i.e., to replacing $\lambda' d(a-b)/2$ by $\lambda' b(c-d)/2$. Thus

$$[Q_L^{rs}(v)]_{\sigma,\sigma'} = \hat{W}_Q(v \mid \sigma_1, \sigma_2, \sigma'_2, \sigma'_1) \cdots \hat{W}_Q(v \mid \sigma_L, \sigma_{L+1}, \sigma'_{L+1}, \sigma'_L), \tag{42}$$

where

$$\hat{W}_Q(v \mid a, b, c, d) = \exp[\lambda' b(c-d)/2 - (\lambda+v)(b-a)(c-d)/4]. \tag{43}$$

This matrix Q_L corresponds to the Q_L of (9.8.27) of ref. 10, but with the normalization changed and both λ and v negated.

This matrix Q_L^{rs} is unchanged by incrementing all the upper spins by unity, and we can define the product

$$I^{rts}(v, v') = Q_L^{rt}(v)\, Q_R^{ts}(v') \tag{44}$$

as above.

Let $\{b_1, \ldots, b_{L+1}\}$ be the spins that are summed over in this matrix product, i.e., the spins on the middle row, above $Q_R^{rt}(v)$ and below $Q_L^{ts}(v')$. They enter only via their differences $\beta_j = b_{j+1} - b_j$. The sum is over all values (± 1) of the β_j, subject to the skew condition

$$\beta_1 + \cdots + \beta_L = t.$$

We can remove this restriction by forming the generating function matrix

$$F^{rs}(z \mid v, v') = \sum_t z^t I^{rts}(v, v') \tag{45}$$

the sum being over all values of t from $-L$ to L, with $L-t$ even. Then $F^{rs}(z \mid v, v')$ is given by the unrestricted sum over the β_j.

Being a transfer matrix product, the elements of $F^{rs}(z \mid v, v')$ are given in the first instance by an expression of type (12). However, now the elements $[M_j]_{c,d}$ of M_j depend on the two intermediate spins c, d only via their difference $m = d - c = \pm 1$. The trace in (12) simplifies to a sum over all the L spin differences. These are now independent, so we can perform the m summation individually for each M_j. The rhs of (12) then becomes a product, as in (10). Hence the elements of $F^{rs}(z\,|\,v, v')$ are given by the rhs of (10), with W replaced by W_F, and

$$W_F(v, v'\,|\,a, b, c, d) = \sum_m z^m \hat{W}_Q(v\,|\,a, b, f+m, f)\, W_Q(v'\,|\,f, f+m, c, d), \tag{46}$$

the sum being over $m=-1$ and $+1$. The rhs is independent of f and from (27), (43)

$$W_F(v, v'\,|\,a, b, c, d) = \sum_m z^m x^{m(b-d)} \mathrm{e}^{m[(\lambda+v)\,m(a-b)+(\lambda+v')(d-c)]/4}. \tag{47}$$

Here v, v' are independent variables.

We look for an auxiliary function $P(v, v'\,|\,a, b)$ such that

$$W_F(v', v\,|\,a, b, c, d) = P(v, v'\,|\,d-a)\, W_F(v, v'\,|\,a, b, c, d)/P(v, v'\,|\,c-b), \tag{48}$$

since then (apart possibly from boundary conditions) the rhs of (10) is unaltered by interchanging v with v', as the P factors cancel out of the product on the rhs. This in turn implies a commutation relation between $Q_L^{rt}(v)$ and $Q_R^{ts}(v')$.

There are four cases to consider in (48), $b-a=\pm 1$ and $c-d=\pm 1$. If $b-a=c-d$, then (48) is automatically satisfied. The other two cases yield just one condition on the function P, which can be written

$$\frac{P(v, v'\,|\,j+1)}{P(v, v'\,|\,j-1)} = \frac{z\,\mathrm{e}^{2v} + x^{2j}\mathrm{e}^{2v'}}{z\,\mathrm{e}^{2v'} + x^{2j}\mathrm{e}^{2v}}\,. \tag{49}$$

Hence $P(j/2)$ has the same structure as the chiral Potts weight function $W_{pq}(j)$ in Eq. (2) of ref. 25, with $\omega = x^4$. However, we emphasize that at this stage we are *not* requiring x^4 to be a root of unity, so the rhs of (49) is not necessarily periodic in j.

Substituting the form (48) for $W=W_F$ into (10), the P-functions cancel except for a boundary factor $P(v, v'\,|\,\sigma_1'-\sigma_1)/P(v, v'\,|\,\sigma_{L+1}'-\sigma_{L+1})$. Remembering that $\sigma_{L+1}=\sigma_1+r$ and $\sigma_{L+1}'=\sigma_1'+s$, this has the form $P(v, v'\,|\,d-a)/P(v, v'\,|\,d-a+s-r)$. This is unity if $s=r$, so we have established the symmetry property $F^{rr}(z\,|\,v', v) = F^{rr}(z\,|\,v, v')$. This is true for *all* z, so the coefficients $I^{rtr}(v, v')$ in the Laurent expansion of $F^{rr}(z\,|\,v', v)$ are also symmetric in v, v', for all t. Hence from (44)

$$Q_L^{rt}(v)\, Q_R^{tr}(v') = Q_L^{rt}(v')\, Q_R^{tr}(v).$$

From (41), replacing t by s, this becomes

$$Q_R^{rs}(v)\, D^r Q_R^{sr}(v') = Q_R^{rs}(v')\, D^r Q_R^{sr}(v). \tag{50}$$

5. SUMMARY OF THE RELATIONS

Neither $Q_R^{rs}(v)$ nor $Q_L^{rs}(v)$ has the desirable property that it is unchanged by incrementing all the spins in *either* the lower or the upper row in Fig. 3 by unity. From the discussion preceding (41), the matrix that does have this property is

$$\tilde{Q}^{rs}(v) = Q_R^{rs}(v)\, D^r = x^{-rs}\, D^{-s} Q_L^{rs}(v). \tag{51}$$

It is convenient to introduce a generalization of the six-vertex model transfer matrix $T^r(v)$:

$$T_s^r(v) = D^{-s} T^r(v)\, D^s. \tag{52}$$

This is the transfer matrix of the six-vertex model with skew parameter r and a field of weight $x^{\sigma_1' - \sigma_1}$ acting on the two vertically adjacent spins (or equivalently the horizontal arrow between them) at the left-hand end of the row. Then the commutation and functional relations (24), (32), (39), and (50) for the T and Q matrices become

$$T_s^r(v)\, T_s^r(v') = T_s^r(v')\, T_s^r(v), \tag{53}$$

$$T_s^r(v)\, \tilde{Q}^{rs}(v') = \tilde{Q}^{rs}(v')\, T_r^s(v), \tag{54}$$

$$T_s^r(v)\, \tilde{Q}^{rs}(v) = \phi(\lambda - v)\, \tilde{Q}^{rs}(v + 2\lambda') + \phi(\lambda + v)\, \tilde{Q}^{rs}(v - 2\lambda'), \tag{55}$$

$$\tilde{Q}^{rs}(v)\, \tilde{Q}^{sr}(v') = \tilde{Q}^{rs}(v')\, \tilde{Q}^{sr}(v) \tag{56}$$

for all values of the skew parameters r, s. Note that nowhere in this paper do we use any implicit summation convention over skew parameters.

Square Matrix Form of the Relations

We have uncovered quite an interesting structure in the T, Q relations for the six-vertex model, and we could continue to work with the above relations (53)–(56). However, they are still not quite in the form of the T, Q (or V, Q) relations given in (9.2.5), (9.8.40), and (9.8.42) of ref. 10. These involve the zero-field six-vertex matrix $T^r(v)$ and a square invertible matrix $Q(v)$, for given r.

To derive these square matrix relations, define, using (56),

$$Q_s^r(v) = \tilde{Q}^{rs}(v)\, \tilde{Q}^{sr}(v_0) = \tilde{Q}^{rs}(v_0)\, \tilde{Q}^{sr}(v) \tag{57}$$

v_0 being some fixed value of v.

Then from (53)–(56) we readily obtain the desired final square matrix relations

$$\begin{aligned} T_s^r(v)\, T_s^r(v') &= T_s^r(v')\, T_s^r(v), \\ T_s^r(v)\, Q_s^r(v') &= Q_s^r(v')\, T_s^r(v), \\ T_s^r(v)\, Q_s^r(v) &= \phi(\lambda - v)\, Q_s^r(v + 2\lambda') + \phi(\lambda + v)\, Q_s^r(v - 2\lambda'), \\ Q_s^r(v)\, Q_s^r(v') &= Q_s^r(v')\, Q_s^r(v) \end{aligned} \tag{58}$$

true for all the allowed values $-L, 2-L, 4-L, \ldots, L$ of r and s.

From our remarks in the next sub-section, we expect $Q_s^r(v)$ to be invertible provided $C_s \geqslant C_r$. For an even number L of columns, this will always be so if we choose $s = 0$. Then $T_0^r(v) = T(v)$ is the usual zero-field six-vertex model transfer matrix and $Q_0^r(v) = Q(v)$ is the associated Q-matrix used in refs. 7–10. We have derived the relations (1)–(4), except that they can be and usually are thought of in the "arrow language" of the next sub-section.

Transformation to Arrow Language

We have used the language of spins on sites to derive these relations, but the T, Q matrices can be thought of conveniently in the alternative spin-difference or "arrow" language, transforming from the σ_j to the α_j by (8). The equations (18), (27), (29), and (51) define the elements (σ, σ') of $\tilde{Q}^{rs}$. They depend only on (α, α'), so we take the elements to be labelled by α, α'. Since the α_j satisfy (9), and the α'_j satisfy (9) with r replaced by s, the matrix $\tilde{Q}^{rs}$ is of dimension C_r by C_s, where

$$C_r = \binom{L}{(L+r)/2}$$

and $Q_s^r(v)$ is a square matrix, of dimension C_r.

From this point of view the six-vertex model transfer matrix is slightly more subtle, since we are not free to increment all the spins in the upper row of Fig. 3 independently of those in the lower row. Only the face configurations of Fig. 6 are allowed. If the spins in the lower row are known, then there two choices for those in the upper, depending on whether

$\sigma_1' = \sigma_1 + 1$ or $\sigma_1' = \sigma_1 - 1$. However, the above matrix products involving T always involve a sum over these two choices. The end result is that the above relations (53)–(56) are also true in the arrow language provided we sum over the two choices of σ_1'. From (18), (23), and (52) we obtain

$$[T_s^r(v)]_{\alpha,\alpha'} = \sum_\beta x^{s\beta} W_{6v}(v\,|\,\sigma_1, \sigma_2, \sigma_2', \sigma_1') \cdots W_{6v}(v\,|\,\sigma_L, \sigma_{L+1}, \sigma_{L+1}', \sigma_L'), \quad (59)$$

the sum being over $\beta = \pm 1$, where $\beta = \sigma_1' - \sigma_1$. The rhs is uniquely defined by $\alpha_1,..., \alpha_L'$. This is the usual six-vertex model transfer matrix in arrow language, with an extra weight x^s (x^{-s}) if the left-hand horizontal arrow points to the left (right). It is of dimension C_r by C_r.

Transformation to all $T_s^r(v)$ Diagonal

We assume that the matrices $T_s^r(v)$, $Q_s^r(v)$ are diagonalizable. Then from the set of relations (58), there exists a non-singular matrix $\mathscr{P}_s^r$, independent of v, such that the transformation $T_s^r(v) \to \mathscr{P}_s^r T_s^r(v) \{\mathscr{P}_s^r\}^{-1}$ reduces both $T_s^r(v)$ and $Q_s^r(v)$ to diagonal form. If we also transform $\tilde{Q}^{rs}(v)$ by $\tilde{Q}^{rs}(v) \to \mathscr{P}_s^r \tilde{Q}^{rs}(v) \{\mathscr{P}_r^s\}^{-1}$, then the relations (53)–(56) and (58) are unchanged.

We do not have a proof for general L, but for $L = 2$ to $L = 6$ we have verified with Mathematica that the matrix product $\tilde{Q}^{rs}(v)\,\tilde{Q}^{sr}(v')$ is invertible if $C_s \geqslant C_r$, for all such values of r and s and for arbitrarily chosen values of v, v', x. This implies that the rectangular matrix $\tilde{Q}^{rs}(v)$ is then of "full" rank, having the maximum rank possible for a matrix of its size:

$$\text{rank of } \tilde{Q}^{rs} = \min(C_r, C_s). \quad (60)$$

From the transformed form of (54) this implies that if $C_s \geqslant C_r$, then the eigenvalues of $T_s^r(v)$ are all contained in the eigenvalues of $T_r^s(v)$, which is an interesting result of which the author was previously not fully aware. If the eigenvalues of $T_r^s(v)$ are non-degenerate, (54) further implies that one can order the eigenvalues so that $\tilde{Q}^{rs}(v)$ transforms to a diagonal (but not necessarily square) matrix. Even if they are degenerate, then the fact that the transform of $Q_s^r(v)$ is also diagonal may imply this property of $\tilde{Q}^{rs}(v)$.

Structure of the Eigenvalues

If $C_s \geqslant C_r$, then it follows from (27) and (29) that for $r \geqslant 0$ all elements of $Q_R^{rs}(v)$ are of the form

$$\mathrm{e}^{v(r-s-L)/4} \times \{\text{polynomial in } \mathrm{e}^v \text{ of degree } (L-r)/2\};$$

while for $r \leqslant 0$ they are of the form

$$e^{v(s-r-L)/4} \times \{\text{polynomial in } e^v \text{ of degree } (L+r)/2\}.$$

If $C_s \leqslant C_r$, then the above statements are true if one interchanges r with s therein. (All four statements are actually true for all r and s, provided one allows some of the initial and/or final coefficients of the polynomial to be zero, which means that some of the v_j below will be infinite.)

From (51) and (57), the same must be true for the elements of $Q^r_s(v)$. Also, since the eigenvector matrices $\mathscr{P}^r_s$ are independent of v, it is true of the diagonalized form of $Q^r_s(v)$, i.e., of its eigenvalues. If $C_s \geqslant C_r$ we can therefore write any eigenvalue as

$$C\, e^{-vs/4} \prod_{j=1}^{(L-r)/2} \sinh \tfrac{1}{2}(v-v_j)$$

if $r \geqslant 0$, or

$$C\, e^{vs/4} \prod_{j=1}^{(L+r)/2} \sinh \tfrac{1}{2}(v-v_j)$$

if $r \leqslant 0$. Here $C, v_1, v_2,\ldots$ are constants, independent of v.

Similarly, the eigenvalues of $T^r_s(v)$ necessarily have the structure

$$C' \prod_{j=1}^{L} \sinh \tfrac{1}{2}(v-w_j),$$

$C', w_1, w_2,\ldots$ being other constants.

We can substitute these forms into (58) and in principle solve for the various constants. Setting $v = v_j$ gives the usual Bethe ansatz equations. Alternatively, one can expand the functions as polynomials in e^v and equate coefficients.

6. A MORE GENERAL FORM FOR *Q*

We remark that (47) is the Boltzmann weight of the "τ_2" model rotated through 90°—Eqs. (3.44) and (3.48) of ref. 14. If instead of (30) we take

$$W_1 = W_{6v}(v), \qquad W_2 = W_F(v', v''), \qquad W_3 = W_F(v'-v-\lambda, v''-v-\lambda),$$

then the star triangle relation (15) remains satisfied, for all v, v', v''.

This is the quite general solution of (15) found in 1990 by Bazhanov and Stroganoff.[13] There are further trivial factors that can be introduced into the Boltzmann weight $W_F(v, v' \mid a, b, c, d)$ without affecting (15), notably $\mu_1^{b-a}\mu_2^{c-d}$, where μ_1, μ_2 are arbitrary constants (the same for both W_2 and W_3). This factor merely multiplies the transfer matrix $F^{rr}(z \mid v, v')$ by $\mu_1^r \mu_2^r$.

From (41), (44), (45), and (51) and the discussion before (50),

$$F^r(v, v') = F^r(v', v) = F^{rr}(z \mid v, v') = \sum_t z^t x^{rt} D^t \tilde{Q}^{rt}(v)\, \tilde{Q}^{tr}(v')\, D^{-t}.$$

It does indeed follow from (52), (55) that

$$T^r(v)\, F^r(v', v'') = F^r(v', v'')\, T^r(v),$$

for all v, v', v''. This is in agreement with the fact that $W_F(v', v'')$ satisfies (15). We stress that this is a "spin language" result: $F^r(v', v'')$ is *not* unchanged by incrementing all the spins in either the lower or upper row of Fig. 3 by unity.

7. CONCLUSION

We have obtained the "T, Q" functional matrix relations for the zero-field six-vertex model (and for other special values of the horizontal field), presenting the working in a way that we hope is more explicit and transparent than those used previously. As a check on our reasoning, we have verified the relations (53)–(56) explicitly on a computer, using Mathematica, for $L = 2$ to $L = 6$.

In some ways this method appears to be less general than Lieb and Sutherland's original Bethe ansatz solutions: it is not obvious how to extend it to the more general situation when there is an arbitrary horizontal electric field (which corresponds to a vertical spin field), or even to solve the zero-field problem with an odd number of columns. Still, it should be remembered that the commuting transfer matrix method was originally developed as a means of solving the eight-vertex model, for which there was then no Bethe ansatz.

We emphasize that *all* the above working is for the general case, when there are no restrictions on the real or complex "crossing parameter" λ.

The "Root of Unity" Cases

A very interesting special case occurs when x^4 is an Nth root of unity, i.e.,

$$\lambda' = \mathrm{i} m \pi / N, \qquad x^{4N} = 1, \tag{61}$$

m, N being integers and k being odd. These cases have been studied by Fabricius and McCoy, and they have shown that the eigenvalues of $T^r(v)$ are then degenerate.[15–18]

One can then regard the lattice spins σ_j, σ'_j as restricted to N values. For a given site these will either be odd or even, e.g. $1, 3, \ldots, 2N-1$. Horizontally adjacent spins differ by ± 1 modulo $2N$, so we continue to take the α_j in (8) to be strictly ± 1, but interpret the difference on the rhs to modulo $2N$. Similarly, the skew parameter r, given by (7), is to be interpreted modulo $2N$. One can then repeat all the above working. (There may have to be some minor modifications, such as multiplying $Q_R^{rs}(v)$ by $\mathrm{e}^{\mathrm{i}\pi r/N}$ so as to ensure that incrementing the spin d in (27) by $2N$ leaves the element of $Q_R^{rs}(v)$ unchanged. The horizontal spin differences $a-b$, $c-d$ therein are to be kept as strictly ± 1.)

There will be one significant difference: in the above we have specialized to the case when no horizontal spin fields (corresponding to vertical arrow or "electric" fields) are applied to the lower or upper rows of the T or Q matrices, for instance including a factor μ^{b-a} in (27). We could have introduced such a factor as it does not affect the star-triangle relation (15), (30). It merely post-multiplies $Q_R^{rs}(v)$ by the diagonal matrix S^z defined in Section 3. Since S^z commutes with $T_s^r(v)$ and $Q_s^r(v)$, the equations (58) would have been unchanged: no increase in generality would have been achieved.

For the root of unity cases this ceases to be true: μ^r is *not* in general the same as μ^{r+2N}. S^z now commutes with $T_s^r(v)$, but *not* with $Q_s^r(v)$. For generality it is therefore important to include such factors in (27). This leads to an extra constant term $\mathscr{C}$ on the rhs of (49). From (48), remembering that $|a-b| = |c-d| = 1$, the function $P(v, v' \mid j)$ is defined only for either j always odd, or j always even. Taking j to be odd, from (49),

$$P(v, v' \mid 2n-1) = \prod_{j=1}^{n} \mathscr{C} \frac{z\,\mathrm{e}^{2v} + \omega^j \mathrm{e}^{2v'}}{z\,\mathrm{e}^{2v'} + \omega^j \mathrm{e}^{2v}}, \tag{62}$$

where

$$\omega = x^4, \qquad \omega^N = 1.$$

The function $P(v, v' \mid j)$ must now be periodic in j of period $2N$, which will be so iff

$$\mathscr{C}^N \{(-z)^N \mathrm{e}^{2Nv} - \mathrm{e}^{2Nv'}\} = (-z)^N \mathrm{e}^{2Nv'} - \mathrm{e}^{2Nv}.$$

With a correct identification of the various parameters, we see that $P(v, v' \mid 2n-1)$ is precisely the N-state chiral Potts model weight function $W_{pq}(n)$.[25] Further, the factor $\mathscr{C}$ arising from the applied fields is important: without it one would have $z = e^{i\pi/N}$ and the model would reduce to the critical Fateev–Zamolodchikov model.[24]

Kashiwara–Miwa Model as a Descendant of the Eight-Vertex Model

Just as the chiral Potts model can be regarded as a "descendant" of the six-vertex model in a vertical arrow field, so Hasegawa and Yamada have shown that the Kashiwara–Miwa model can be regarded as a "descendant" of the zero-field eight-vertex model.[22, 23] Note that neither vertex model includes the other: they intersect in the zero-field six-vertex model. Correspondingly, the chiral Potts and Kashiwara model intersect in the Fateev–Zamolodchikov model.

In the same way that our present approach leads very directly from the six-vertex model to the chiral Potts model, so does it lead from the eight-vertex model to the Kashiwara–Miwa model. For the eight-vertex model, the function $W_F(u, v \mid a, b, c, d)$ is the multiplicand of eqn. (10.5.27) of ref. 10, the $s_j, s'_j, \sigma_j, \sigma'_j$ therein being

$$s_j = s + \lambda d, \qquad s'_j = s' + \lambda a, \qquad \sigma_j = c - d, \qquad \sigma'_j = b - a, \tag{63}$$

and s, s' are arbitrary parameters.[7]

Here we established the Q, Q commutation by showing that $F^{rr}(z \mid v, v') = F^{rr}(z \mid v', v)$. Similarly, in ref. 10 the author established the relation (10.5.29) by showing that the product (10.5.27) is a symmetric function of v and v'. That argument was by inductive reasoning on the entire product. If instead we look for a solution for the function P in (48), we find that the P-factors depend on a, d (or b, c) not just via their difference, but now $P(v, v' \mid a, d)$ is a product of a function of $d-a$ difference and a function of $a+d$:

$$P(v, v' \mid a, d) = P_1(v, v' \mid d-a)\, P_2(v, v' \mid a+d).$$

Focussing on the root of unity case, when $\lambda' = \lambda - 2iK = 2im\, K/N$, m and N being integers, and requiring that $P(v, v' \mid a+2N, d) = P(v, v' \mid a, d+2N) = P(v, v' \mid a, d)$, we find that $P(v, v' \mid a, d)$ is the Boltzmann weight function of

[7] Eq. (10.5.8) of ref. 10 contains an error: σ_{j+1} therein should be σ_{j-1}.

the Kashiwara–Miwa model. Since the eight-vertex model transfer matrix does not commute with S^z, it does not break up into diagonal blocks with different r and there is no analogue of the arbitrary factor $\mathscr{C}$ in (62).

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